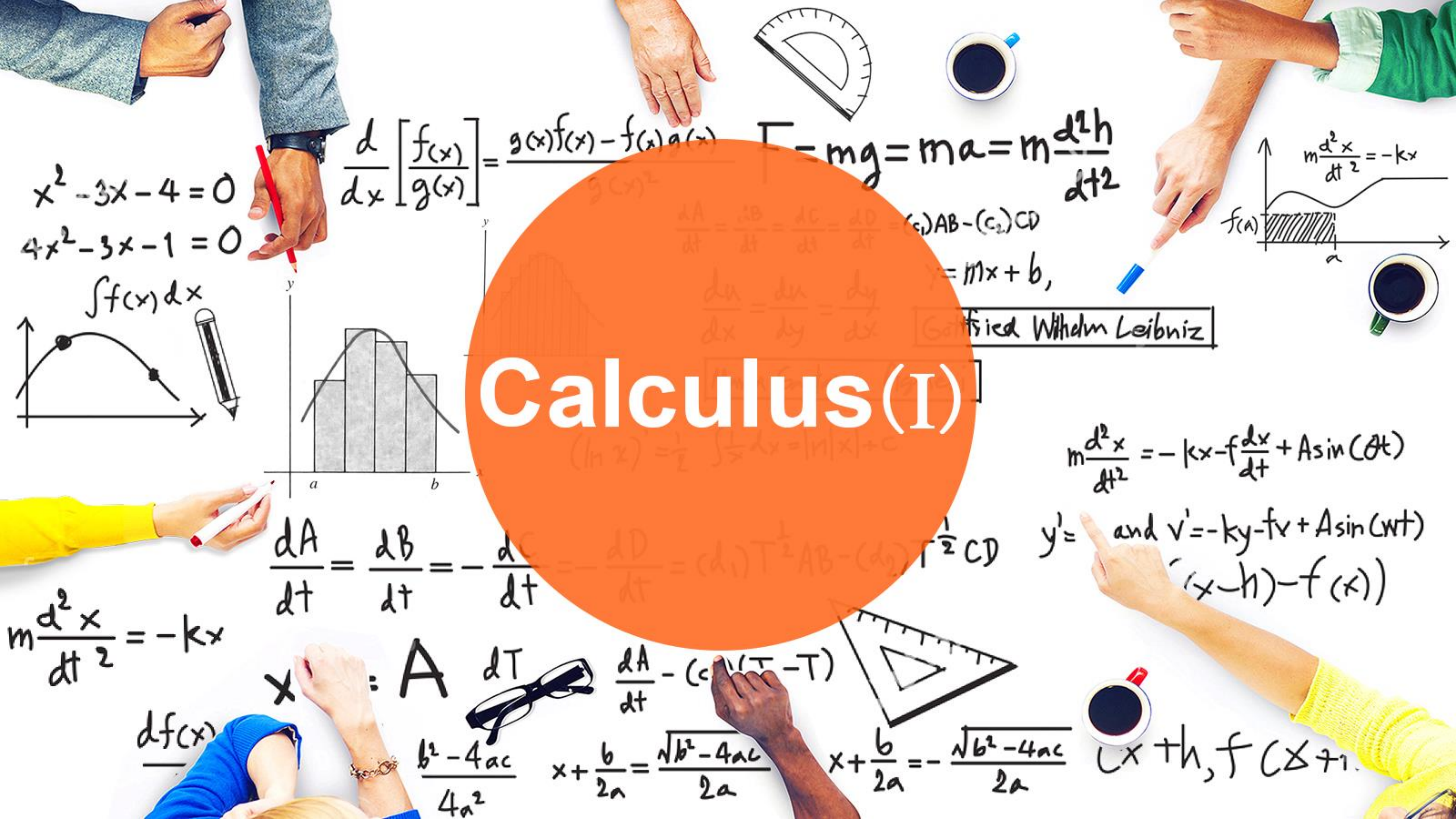
Calculus(Ⅰ)
x² - 3x - 4 = 0
4x² - 3x - 1 = 0
∫f(x) dx
F = mg = ma = m d²h/dt²
Gottfried Wilhelm Leibniz
m d²x/dt² = -kx
m d²x/dt² = -kx - f dx/dt + Asin(θt)
y' = v and v' = -ky - fv + Asin(wt)
x + b/2a = √(b² - 4ac)/2a
x + b/2a = - √(b² - 4ac)/2a

4.5 The Mean Value Theorem for Integrals and the Use of Symmetry

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Definition of Average Value of a Function

If f is integrable on the interval $[a, b]$, then the **average value** of f on $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)dx$$

Example 1

Find the average value of the function defined by $f(x) = x \sin x^2$ on the interval $[0, \sqrt{\pi}]$. (See Figure 1 in the textbook.)

The average value is $\dfrac{1}{\sqrt{\pi} - 0} \displaystyle\int_0^{\sqrt{\pi}} x \sin x^2 dx$

To evaluate this integral, we make the substitution $u = x^2$, so that $du = 2xdx$.

When $x = 0$, $u = 0$ and when $x = \sqrt{\pi}$, $u = \pi$. Thus,

$$\frac{1}{\sqrt{\pi}} \int_0^{\sqrt{\pi}} x \sin x^2 dx = \frac{1}{\sqrt{\pi}} \int_0^{\pi} \frac{1}{2} \sin u \, du = \frac{1}{2\sqrt{\pi}} [-\cos u]_0^{\pi} = \frac{1}{2\sqrt{\pi}} (2) = \frac{1}{\sqrt{\pi}}$$

Theorem A: Mean Value Theorem of Integrals

If f is continuous on $[a, b]$, then there is a number c between a and b such that

$$f(c) = \frac{1}{b-a}\int_a^b f(t)dt$$

Proof: For $a \leq x \leq b$ define $G(x) = \int_a^x f(t)dt$.

By the Mean Value Theorem for Derivatives (applied to G) there is a c in the interval (a, b) such that $G'(c) = \frac{G(b)-G(a)}{b-a}$

Since $G(a) = \int_a^a f(t)dt = 0$, $G(b) = \int_a^b f(t)dt$, and $G'(c) = f(c)$, this leads to $G'(c) = f(c) = \frac{1}{b-a}\int_a^b f(t)dt$

Example 2

Some examples of definite integrals for odd, even functions and periodic functions

Prove that if $f(x)$ is integrable on the interval $[-a, a]$, then

$$\int_{-a}^{a} f(x)\mathrm{d}x = \int_{0}^{a}[f(x) + f(-x)]\mathrm{d}x$$

$$\int_{-a}^{a} f(x)\mathrm{d}x = \boxed{\int_{-a}^{0} f(x)\mathrm{d}x +} \int_{0}^{a} f(x)\mathrm{d}x$$

Let $x = -t$, $\mathrm{d}x = -\mathrm{d}t$.

Then, $x = -a$, $t = a$; $x = 0$, $t = 0$.

$$\int_{-a}^{0} f(x)\mathrm{d}x = -\int_{a}^{0} f(-t)\mathrm{d}t = \int_{0}^{a} f(-x)\mathrm{d}x$$

$$\therefore \quad \int_{-a}^{a} f(x)\mathrm{d}x = \int_{0}^{a}[f(x) + f(-x)]\mathrm{d}x$$

Example 3

Find $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos x}{1+e^{-x}} \mathrm{d}x$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos x}{1+e^{-x}} \mathrm{d}x = \int_{0}^{\frac{\pi}{4}} \left[\frac{\cos x}{1+e^{-x}} + \frac{\cos x}{1+e^{x}} \right] \mathrm{d}x$$

$$= \int_{0}^{\frac{\pi}{4}} \cos x \mathrm{d}x$$

$$= \frac{\sqrt{2}}{2}$$

Theorem B

available from $\int_{-a}^{a} f(x)\mathrm{d}x = \int_{0}^{a}[f(x)+f(-x)]\mathrm{d}x$

Symmetry Theorem

If $f(x)$ is continuous on the interval $[-a, a]$, and

(1) $f(x)$ is an even function, then $\int_{-a}^{a} f(x)\mathrm{d}x = 2\int_{0}^{a} f(x)\mathrm{d}x$

(2) $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\mathrm{d}x = 0$

Some Examples

$$\int_{-\pi}^{\pi} x^4 \sin x \mathrm{d}x = 0$$

$$\int_{-5}^{5} \frac{x^3 \sin^2 x}{x^4 + 2x^2 + 1} \mathrm{d}x = 0$$

$$\int_{-1}^{1} \sqrt{4 - x^2} \mathrm{d}x = 2\int_0^1 \sqrt{4 - x^2} \mathrm{d}x$$

Odd Function **Even Function**

$$\int_{-2}^{2} \frac{x^5 + x^4 - x^3 - x^2 - 2}{1 + x^2} \mathrm{d}x = \int_{-2}^{2} \frac{x^5 - x^3}{1 + x^2} \mathrm{d}x + \int_{-2}^{2} \frac{x^4 - x^2 - 2}{1 + x^2} \mathrm{d}x$$

$$= 0 + 2\int_0^2 \frac{x^4 - x^2 - 2}{1 + x^2} \mathrm{d}x = -\frac{8}{3}$$

Example 4

If $f(x)$ is continuous on the interval $[0, 1]$, prove that

(1) $\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\cos x)\mathrm{d}x;$

(2) $\int_0^{\pi} xf(\sin x)\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\mathrm{d}x,$

and find $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\mathrm{d}x.$

The definite integral formula of trigonometric functions

Example 4

$$(1)\ \int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\cos x)\,\mathrm{d}x;$$

Let $x = \frac{\pi}{2} - t \Rightarrow \mathrm{d}x = -\mathrm{d}t$

$$\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x = -\int_{\frac{\pi}{2}}^{0} f\left[\sin\left(\frac{\pi}{2} - t\right)\right]\mathrm{d}t$$

$$= \int_0^{\frac{\pi}{2}} f(\cos t)\,\mathrm{d}t$$

$$= \int_0^{\frac{\pi}{2}} f(\cos x)\,\mathrm{d}x$$

Example 4

(2) $\int_0^{\pi} x f(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,\mathrm{d}x$, find $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x.$

Let $x = \pi - t \implies \mathrm{d}x = -\mathrm{d}t$

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,\mathrm{d}x$$

$$= -\frac{\pi}{2}\int_0^{\pi} \frac{1}{1+\cos^2 x}\,\mathrm{d}(\cos x)$$

$$= -\frac{\pi}{2}\left[\arctan(\cos x)\right]_0^{\pi}$$

$$= -\frac{\pi}{2}\left(-\frac{\pi}{4} - \frac{\pi}{4}\right) = \frac{\pi^2}{4}.$$

Theorem C

If f is periodic with period p, then $\int_{a+p}^{b+p} f(x)dx = \int_a^b f(x)dx$

Proof: The geometric interpretation can be seen in the textbook.

To prove the result, let $u = x - p$ so that $x = u + p$ and $du = dx$. Then

$$\int_{a+p}^{b+p} f(x)dx = \int_a^b f(u+p)du = \int_a^b f(u)du = \int_a^b f(x)dx$$

We could replace $f(u+p)$ by $f(u)$ because f is periodic.

Example 5

Let $f(x)=\begin{cases}1+x^2, & x<0,\\ e^{-x}, & x\geq 0,\end{cases}$ Find $\int_1^3 f(x-2)\mathrm{d}x.$

Let $x-2=t,\quad x=2+t \Rightarrow \mathrm{d}x=\mathrm{d}t$

$$\int_1^3 f(x-2)\mathrm{d}x=\int_{-1}^{1} f(t)\,\mathrm{d}t$$

$$=\int_{-1}^{0}(1+t^2)\mathrm{d}t+\int_0^1 e^{-t}\mathrm{d}t$$

$$=\frac{7}{3}-\frac{1}{e}$$

Example 6

Find $\lim\limits_{x\to 0}\frac{1}{x^2}\int_0^x \frac{\sin xt}{t}\mathrm{d}t.$

Let $xt = u$, Then $t = \frac{u}{x}$, $\mathrm{d}t = \frac{\mathrm{d}u}{x}$.

$t = 0 \Rightarrow u = 0$; $t = x \Rightarrow u = x^2$.

$$\int_0^x \frac{\sin xt}{t}\mathrm{d}t = \int_{\mathbf{0}}^{\mathbf{x^2}} \frac{\sin u}{\frac{u}{x}}\cdot\frac{\mathrm{d}u}{x} = \int_0^{x^2}\frac{\sin u}{u}\mathrm{d}u$$

$$\therefore L = \lim_{x\to 0}\frac{\int_0^{x^2}\frac{\sin u}{u}\mathrm{d}u}{x^2}\left(\frac{\mathbf{0}}{\mathbf{0}}\right) = \lim_{x\to 0}\frac{\frac{\sin x^2}{x^2}\cdot 2x}{2x} = \mathbf{1}$$

The Mean Value Theorem for Integrals and the Use of Symmetry